Int. J. Solids Structures Vol. 23, No. 1, pp. 45-55, 1987
Printed in Great Britain.
0020-7683/87 $3.00+.00
Pergamon Journals Ltd.

VIBRATION AND BUCKLING OF RECTANGULAR PLATES WITH NONUNIFORM ELASTIC CONSTRAINTS IN ROTATION

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(*Received* 25 *July* 1985; *in revised form* 8 *February* 1986)

Abstract—This paper deals with vibration and buckling analyses of rectangular plates with nonuniform elastic constraints against rotation using the spline strip method. The effect of nonuniformity of elastic constraints and aspect ratios on these problems is analyzed, and the results are also compared with those based on uniform elastic constraints. It is found from the numerical examples that the influence of the nonuniformity of elastic constraints on the natural frequencies and buckling loads become evident.

1. INTRODUCTION

Extensive study has been carried out on the analyses of vibration[1-3] and buckling[4] of plates with classical boundary conditions. However the boundary conditions in many structures which are used in civil, mechanical, ship or aeronautical engineering are far from classical in nature. Such edges may be considered as elastically restrained in rotation and/or deflection.

Vibration[5-14] and buckling[15-18] of plates restrained against rotation along edges have been analyzed utilizing numerical methods such as the Rayleigh-Ritz method, Galerkin method and finite element method. However, all of the foregoing references consider only elastic constraints which are uniform along a given boundary. Elastic constraints represent stiffness coupling with surrounding support structure and the stiffness of such surrounding structure along the common boundary may not be constant in practical applications, but will vary from point to point.

Recently Leissa *et al.* analyzed the vibration problem for nonuniform elastic translation and/or rotational constraints in the case of a circular plate by using the exact method[19]. Laura *et al.*[20-22] obtained the solution of vibration and buckling of circular plates with nonuniform edge constraints by the Rayleigh-Ritz method. Moreover Leissa *et al.*[23] analyzed vibrations of rectangular plates with nonuniform elastic edge supports in rotation using the exact method and the Rayleigh-Ritz method.

Fujii and Hoshino[24] presented discrete and non-discrete mixed method with B-spline functions to analyze vibration and buckling of rectangular plates. Fan and Cheung[25] applied the spline finite strip method to analyze vibrations of rectangular plates with complex support conditions, and used spline functions to replace the function series which satisfies *a priori* the end conditions of a strip.

In this paper, vibration and buckling of isotropic thin rectangular plates with nonuniform and continuous constraints against rotation are analyzed by using the spline strip method[26] which is a refined finite strip approach with displacement functions given as the product of Fourier series in the longitudinal direction and spline functions in the other direction.

The effect of nonuniform elastic constraints in rotation on frequencies and buckling loads of rectangular plates are investigated.

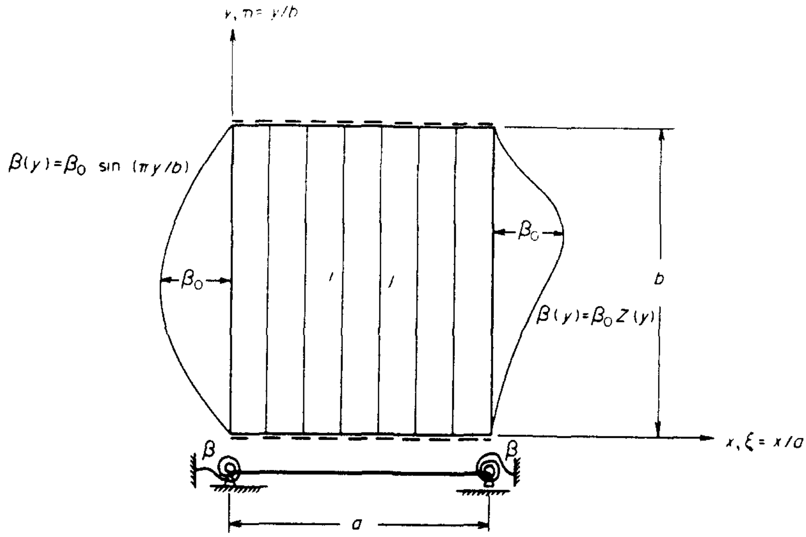

Fig. 1. Rectangular plate with nonuniform elastic constraints in rotation and the definition of the distribution function of nonuniform constraints.

2. FORMULATION BY SPLINE STRIP METHOD

The solution procedure of vibration and buckling of rectangular plates with elastically nonuniform restraints against rotation using the spline strip method is described in this section. The rectangular plate is idealized by discrete strip elements as shown in Fig. 1.

The displacement function is expressed by the product of the basic function series in the longitudinal direction and spline functions which are known as piecewise polynomials in the other direction

$$W(x, y) = \sum_{m=1}^{r} S(x)Y_m(y) \tag{1}$$

in which $Y_m(y)$ is the basic function satisfying the end conditions in the y-direction. The spline function, $S(x)$, of which high derivatives are continuous in the discretized sub-regions and higher numerical stability, is obtained. The spline functions[27] are defined as

$$S(x) = \sum_{n=1}^{i_x} C_n N_{n,k}(x), \qquad i_x = k + M_x - 1 \tag{2}$$

where $N_{n,k}(x)$ is the normalized B-spline functions, $k - 1$ is the degrees of B-spline functions and M_x is the number of strips.

Substituting eqn (2) into eqn (1), the displacement function is expressed by

$$W = \sum_{m=1}^{r} \sum_{n=1}^{i_x} C_{nm} N_{n,k}(x) Y_m(y) \tag{3}$$

or

$$W = \sum_{m=1}^{r} [N] Y_m \{C\}_m \tag{4}$$

where

$$[N] = [N_{1,k}(x), N_{2,k}(x), \ldots, N_{i_x,k}(x)] \tag{5}$$

and

$$\{C\}_m^T = \{C_{1m} C_{2m} C_{3m}, \ldots, C_{i_x m}\}. \tag{6}$$

The strain energy due to bending U_p, the kinetic energy T of the isotropic rectangular plate and the potential V of the in-plane loads S_x, S_y and S_{xy} are given in the Cartesian coordinate system as follows:

$$U_p = (D/2)(b/a^3)\int_0^1\int_0^1 [(\partial^2 W/\partial\xi^2 + \lambda^2\partial^2 W/\partial\eta^2)^2 - 2(1-\nu)\lambda^2\{(\partial^2 W/\partial\xi^2)(\partial^2 W/\partial\eta^2) - (\partial^2 W/\partial\xi\partial\eta)^2\}]\,d\xi\,d\eta \tag{7}$$

$$T = (1/2)\rho hab\omega^2\int_0^1\int_0^1 W^2\,d\xi\,d\eta \tag{8}$$

in which $\xi = x/a$, $\eta = y/b$, $\lambda = a/b$, ν is Poisson's ratio, D is flexural rigidity, h is thickness, ρ is density and ω is circular frequency (rad s^{-1}) of the rectangular plate.

$$V = -(1/2)\int_0^1\int_0^1 [N_x(\partial W/\partial\xi)^2 + 2\lambda N_{xy}(\partial W/\partial\xi)(\partial W/\partial\eta) + \lambda^2 N_y(\partial W/\partial\eta)^2]\,d\xi\,d\eta \tag{9}$$

where $N_x = S_x\,(bh/a)$, $N_{xy} = S_{xy}(bh/a)$ and $N_y = S_y(bh/a)$.

To deal with nonuniform elastic constraints against rotation along the edges as shown in Fig. 1, some translational and rotational elastic springs having stiffnesses α and β, respectively, are attached to the edges. Here $\alpha(y)$ and $\beta(y)$ along the edges at $x = 0$ and a are defined by

$$\alpha(y) = \alpha_0, \qquad \beta(y) = \beta_0 Z(y) \tag{10}$$

in which $Z(y)$ shows the distribution function corresponding to nonuniform elastic constraints, and α_0 and β_0 are the maximum values of the elastic springs. Hence, the following boundary conditions are required along the constrained edges

$$\left.\begin{aligned} V_n(s) &= \alpha W \\ M_n(s) &= \beta(\partial W/\partial n) = \beta_0 Z(s)(\partial W/\partial n) \end{aligned}\right\}. \tag{11}$$

The strain energy due to the continuous elastic constraints, U_b is given by

$$U_b = (1/2)\oint M_n(s)(\partial W/\partial n)\,ds + (1/2)\oint V_n(s) W\,ds \tag{12}$$

in which V_n is the edge reaction, M_n is the normal bending moment and $\partial W/\partial n$ is the normal slope at the edges. The coordinate along the boundary of the edges is s.

The total strain energy U including that of the rectangular plate and the energy due to elastic constraints is presented as

$$U = U_p + U_b. \tag{13}$$

The displacement functions defined previously are expressed as follows:

$$\left.\begin{aligned} W_{mn}(\xi,\eta) &= \sum_{m=1}^{r}\sum_{n=1}^{i_x} C_{nm} N_{n,k}(\xi) Y_m(\eta) \\ W_{sd}(\xi,\eta) &= \sum_{s=1}^{r}\sum_{d=1}^{i_x} C_{ds} N_{d,k}(\xi) Y_s(\eta). \end{aligned}\right\} \tag{14}$$

By substituting eqns (14) into eqns (13), (8) and (9), the expressions for U, T and V become

$$\begin{aligned} U = {} & (D/2)(b/a^3) \sum_{m=1}^{r}\sum_{n=1}^{i_x}\sum_{s=1}^{r}\sum_{d=1}^{i_x} C_{nm}C_{ds}[I_{nd}^{22}J_{ms}^{00} + \lambda^4 I_{nd}^{00}J_{ms}^{22} \\ & + \lambda^2\{2(1-\nu)I_{nd}^{11}J_{ms}^{11} + \nu(I_{nd}^{02}J_{ms}^{20} + I_{nd}^{20}J_{ms}^{02})\}] \\ & + (1/2)\sum_{m=1}^{r}\sum_{n=1}^{i_x}\sum_{s=1}^{r}\sum_{d=1}^{i_x} C_{nm}C_{ds}[\{\alpha_0 N_{nd}^{00}J_{ms}^{00} + (\beta_0/a^2)N_{nd}^{11}L_{ms}^{00}\}|_{\xi=0} \\ & + \{\alpha_0 N_{nd}^{00}J_{ms}^{00} + (\beta_0/a^2)N_{nd}^{11}L_{ms}^{00}\}|_{\xi=1}] \end{aligned} \tag{15}$$

$$T = (1/2)\rho h\omega^2 ab \sum_{m=1}^{r}\sum_{n=1}^{i_x}\sum_{s=1}^{r}\sum_{d=1}^{i_x} C_{nm}C_{ds}I_{nd}^{00}J_{ms}^{00} \tag{16}$$

and

$$\begin{aligned} V = {} & (1/2)\sum_{m=1}^{r}\sum_{n=1}^{i_x}\sum_{s=1}^{r}\sum_{d=1}^{i_x} C_{nm}C_{ds}[N_x I_{nd}^{11}J_{ms}^{00} + 2\lambda N_{xy}I_{nd}^{10}J_{ms}^{01} \\ & + \lambda^2 N_y I_{nd}^{00}J_{ms}^{11}] \end{aligned} \tag{17}$$

in which the integrals I_{nd}^{pq}, J_{ms}^{pq} and L_{ms}^{pq} are defined by

$$\left.\begin{aligned} I_{nd}^{pq} &= \int_0^1 N_{n,k}^{(p)}(\xi)N_{d,k}^{(q)}(\xi)\,\mathrm{d}\xi \\ J_{ms}^{pq} &= \int_0^1 Y_m^{(p)}(\eta)Y_s^{(q)}(\eta)\,\mathrm{d}\eta \\ L_{ms}^{pq} &= \int_0^1 Z(\eta)Y_m^{(p)}(\eta)Y_s^{(q)}(\eta)\,\mathrm{d}\eta \end{aligned}\right\} \tag{18}$$

and N_{nd}^{pq} is expressed as

$$N_{nd}^{pq} = N_{n,k}^{(p)}(\xi)N_{d,k}^{(q)}(\xi) \tag{19}$$

where p and q are the order of the derivative of the basic functions or B-spline functions.

The coefficients $\{C\}$ are determined by using the principle of minimum potential energy as follows:

$$\partial(U - T - V)/\partial C_{nm} = 0, \qquad \begin{cases} n = 1, 2, \ldots, i_x \\ m = 1, 2, \ldots, r \end{cases} \tag{20}$$

which can be expressed in matrix form

$$[K]\{C_{sd}\} = n^{*2}[M] + k^*[G]. \tag{21}$$

Here $n^* = (\omega a^2/\pi^2)\sqrt{(\rho h/D)}$ and k^* is the buckling load parameter. $[K]$, $[M]$ and $[G]$ are matrices obtained from eqns (15), (16) and (17), respectively. The order of these matrices is expressed by $(k + M_x - 1)xr$, where $k - 1$ is the degrees of B-spline functions, M_x is the number of strips and r is the number of series terms in the basic functions.

3. NUMERICAL EXAMPLES AND DISCUSSIONS

The authors[26] show the application of the spline strip method to analyze bending and vibration of skew plates with classical boundary conditions, and good accuracy and stable convergence of the results have been obtained.

In this paper, the effects of nonuniformity of elastically continuous constraints against rotation on natural frequencies and buckling loads of rectangular plates are studied.

In the numerical examples, the basic function Y_m is defined as

$$Y_m(y) = \sin(m\pi y/b). \tag{22}$$

The distribution function of elastic constraints against rotation is assumed as follows at $x = 0$ and/or $x = a$

$$Z(y) = \sin(\pi y/b). \tag{23}$$

To evaluate the rotational stiffness of elastic springs, the rotational stiffness parameter, κ is introduced as

$$\kappa = \beta_0 b/D \tag{24}$$

where β_0 is the maximum stiffness of elastically rotational springs and b is the length of edges as shown in Fig. 1.

The plate under consideration is rigidly supported against transverse displacement: $\alpha_0 = \infty$ is assumed in calculations.

To demonstrate the accuracy and convergence studies of the present approach, vibration and buckling of skew plates with uniform elastic constraints in rotation have been analyzed, and good accuracy and stable convergence were obtained in Refs [26, 28].

3.1. *Vibration of rectangular plates with elastically restrained edges against rotation*

To show the accuracy of the present numerical method, fundamental frequency parameters, $\bar{n}_1^* = \omega a^2\sqrt{(\rho h/D)}$ of isotropic rectangular plates with nonuniform rotational constraints on two opposite edges are presented in Table 1, comparing with the exact solutions and the approximate values obtained using the Ritz method by Leissa *et al.*[23]. Here, the continuous rotational stiffnesses are assumed to vary along two opposite edges by $\beta(x) = \beta_0 x(a - x)$.

Table 1. Comparison of fundamental frequency parameters, $\bar{n}_1^* = \omega a^2\sqrt{(\rho h/D)}$ for simply supported rectangular plates with parabolically varying rotational constraints on two opposite edges, $\beta(x) = \beta_0 x(a-x)$, $\nu = 0.3$ and $\alpha_0 = \infty$

$\lambda=a/b$	$\kappa=\beta_0 a^3/D$	present method*	series solution [23]	Ritz method [23]
0.5	0.0	12.337	12.337	-
	0.1	12.341	12.341	12.344
	1.0	12.373	12.372	12.401
	10.0	12.626	12.621	12.975
	100.0	13.333	13.319	13.598
	∞	13.688	13.688	-
1.0	0.0	19.739	19.739	-
	0.1	19.758	19.757	19.765
	1.0	19.924	19.915	20.114
	10.0	21.249	21.235	24.068
	100.0	25.815	25.799	28.193
	∞	28.951	28.951	-
2.0	0.0	49.348	49.348	-
	0.1	49.415	49.405	49.428
	1.0	50.064	49.914	50.981
	10.0	54.423	54.387	71.545
	100.0	74.546	74.223	91.890
	∞	95.256	95.256	-

* In the present calculations, $k-1=4$, $M_x=8$ and $r=15$ are used.

It is found from the table that present results are closer to the exact solutions[23] than those of the Ritz method for different aspect ratios and rotational rigidity parameters, $\beta_0 a^3/D$.

To investigate the influence of the continuous distribution of nonuniform elastic constraints in rotation on the natural frequency parameters, $n^* = (\omega/\pi^2)a^2\sqrt{(\rho h/D)}$, several numerical examples are carried out. Here the nonuniform elastic constraint is assumed to be distributed as the half sine function (see Fig. 1), and the effect of in-plane load is neglected ($S_y = 0$). In all calculations, the number of strips, $M_x = 8$, the terms of Fourier series, $r = 15$ and the degree of spline functions $k - 1 = 4$ are used.

Table 2 shows the first three frequency parameters of simply supported rectangular plates with opposite edges elastically restrained against rotation. The rotational stiffness parameter, $\kappa = \beta_0 b/D$ varies from 0 to ∞, and aspect ratios of $\lambda = a/b$ of 0.5, 1.0 and 2.0 are used. The results for plates with nonuniform elastic constraints are also compared with those for uniform constraints having the same maximum stiffness, β_0 of rotational springs and with those for uniform average constant value, $(2/\pi)\beta_0$ which gives the same area under the curve of $\beta(y) = \beta_0 \sin(\pi y/b)$.

It is found that the frequency parameters are influenced by the distribution of elastically rotational constraints. The effect of rotational parameters on the frequencies is evidently shown for lower aspect ratios. The results calculated for the case of nonuniform rotational constraints always show values between those in the cases of uniform constant constraints, β_0 and of uniform average constant values, $(2/\pi)\beta_0$.

Table 3 shows the first three frequency parameters of rectangular plates with two opposite edges elastically restrained in rotation and free, respectively (see Type 3 in Fig. 2).

Table 2. The first three natural frequency parameters, $n^* = (\omega/\pi^2)a^2\sqrt{(\rho h/D)}$ of rectangular plates with two opposite edges elastically constrained in rotation; $M_x = 8$, $k - l = 4$, $r = 15$, $\nu = 0.3$, $\alpha_0 = \infty$ and $S_y = 0$

$\kappa=\beta_0 b/D$	Modes	Aspect ratio, $\lambda=a/b$ 0.5 †	0.5 ††	0.5 †††	1.0 †	1.0 ††	1.0 †††	2.0 †	2.0 ††	2.0 †††
0.0	1	5.000	5.000	5.000	2.000	2.000	2.000	1.250	1.250	1.250
	2	8.000	8.000	8.000	5.000	5.000	5.000	2.000	2.000	2.000
	3	13.00	13.00	13.00	5.000	5.000	5.000	3.250	3.250	3.250
1.0	1	5.257	5.197	5.300	2.079	2.060	2.091	1.264	1.262	1.267
	2	8.133	8.125	8.192	5.026	5.025	5.038	2.035	2.030	2.040
	3	13.08	13.08	13.12	5.129	5.098	5.150	3.307	3.293	3.314
5.0	1	6.034	5.830	6.176	2.291	2.239	2.327	1.303	1.294	1.308
	2	8.574	8.548	8.794	5.109	5.105	5.148	2.139	2.115	2.155
	3	13.35	13.35	13.51	5.511	5.411	5.579	3.458	3.420	3.483
10.0	1	6.570	6.397	6.863	2.442	2.380	2.484	1.326	1.317	1.332
	2	8.987	8.955	9.307	5.182	5.176	5.233	2.206	2.179	2.224
	3	13.62	13.62	13.87	5.813	5.684	5.902	3.568	3.523	3.598
20.0	1	7.431	7.132	7.646	2.598	2.539	2.637	1.347	1.340	1.353
	2	9.547	9.518	9.935	5.271	5.266	5.327	2.271	2.247	2.286
	3	14.01	14.01	14.33	6.157	6.024	6.249	3.681	3.638	3.709
40.0	1	8.171	7.911	8.364	2.727	2.683	2.757	1.364	1.358	1.367
	2	10.17	10.17	10.55	5.360	5.358	5.410	2.321	2.304	2.332
	3	14.47	14.50	14.82	6.470	6.361	6.546	3.775	3.743	3.796
100.0	1	8.904	8.746	9.030	2.837	2.814	2.854	1.376	1.374	1.378
	2	10.88	10.89	11.15	5.451	5.452	5.482	2.362	2.354	2.368
	3	15.04	15.10	15.32	6.756	6.695	6.802	3.853	3.838	3.865
∞	1	9.652	9.652	9.652	2.933	2.933	2.933	1.387	1.387	1.387
	2	11.73	11.73	11.73	5.546	5.546	5.546	2.396	2.396	2.396
	3	15.84	15.84	15.84	7.024	7.024	7.024	3.921	3.921	3.921

† : $\beta(y)=\beta_0 \sin(\pi y/b)$, †† : $\beta(y)=(2/\pi)\beta_0$ and ††† : $\beta(y)=\beta_0$

The values calculated for the case of nonuniform constraints are less than those in the case of uniform constraints, and the difference between them becomes larger with a decrease of aspect ratios.

3.2. *Buckling of rectangular plates with elastically restrained edges against rotation*

The loaded edges are considered to be simply supported, while the unloaded edges are either restrained in rotation, or are completely free both laterally and rotationally.

In the calculations, three types of rectangular plates with several constraints as shown in Fig. 2 are analyzed by the present method, and the results are also compared with those calculated in the case of uniform constraints in rotation.

Table 4 shows the influence of nonuniform rotational constraints on the buckling load parameter, $k^* = S_y a^2/D\pi^2$ of rectangular plates with opposite edges elastically constrained against rotation (Type 1), subjected to uniform compression S_y. The rotational stiffness parameter $\kappa = \beta_0 b/D$ varies from 0 to ∞, and the aspect ratios of $\lambda' = b/a$ of 0.5, 1.0, 1.5, 2.0 and 2.5 are used. The effect of nonuniformity of the elastic constraints on this problem is shown in Fig. 3 for the different aspect ratios and rotational stiffness parameters.

It is found that the effect of nonuniformity of rotational constraints on the buckling load parameters is concerned with the aspect ratios and rotational stiffness parameters. However the influence of the nonuniformity of the constraints is deduced with increasing of the rotational stiffness parameters.

Tables 5 and 6 show the buckling load parameters of rectangular plates with opposite edges simply supported and elastically restrained against rotation (Type 2), and with opposite edges elastically restrained and free (Type 3), respectively, subjected to uniform compression S_y.

It is seen that the effect of nonuniform elastic constraints on the results depends on

Table 3. The first three natural frequency parameters, n^* of rectangular plates with opposite edges elastically constrained in rotation and free; $M_x = 8$, $k - 1 = 4$, $r = 15$, $S_y = 0$ and $\nu = 0.3$

$\kappa = \beta_0 b/D$	Modes	Aspect ratio, $\lambda = a/b$					
		0.5		1.0		2.0	
		†	††	†	††	†	††
0.0	1	1.635	1.635	1.184	1.184	0.3300	0.3300
	2	4.735	4.735	2.812	2.812	0.4731	0.4731
	3	7.628	7.628	4.175	4.175	0.7568	0.7568
1.0	1	1.698	1.708	1.196	1.197	1.045	1.045
	2	4.756	4.764	2.859	2.867	1.508	1.510
	3	7.758	7.781	4.177	4.177	2.415	2.419
5.0	1	1.863	1.824	1.224	1.224	1.049	1.055
	2	4.820	4.843	2.986	3.039	1.537	1.535
	3	8.170	8.578	4.186	4.207	2.472	2.486
10.0	1	1.975	2.005	1.241	1.246	1.051	1.052
	2	4.875	4.913	3.075	3.099	1.554	1.558
	3	8.516	8.623	4.194	4.199	2.509	2.519
20.0	1	2.086	2.114	1.257	1.261	1.053	1.054
	2	4.941	4.984	3.165	3.188	1.569	1.573
	3	8.935	9.054	4.202	4.207	2.545	2.554
40.0	1	2.175	2.196	1.269	1.272	1.055	1.055
	2	5.008	5.043	3.238	3.255	1.581	1.583
	3	9.337	9.443	4.211	4.214	2.573	2.579
60.0	1	2.214	2.230	1.274	1.276	1.055	1.055
	2	5.040	5.071	3.269	3.283	1.585	1.587
	3	9.533	9.624	4.214	4.218	2.584	2.589
80.0	1	2.235	2.248	1.276	1.278	1.055	1.055
	2	5.059	5.087	3.288	3.299	1.588	1.589
	3	9.649	9.727	4.217	4.219	2.591	2.595
100.0	1	2.249	2.260	1.278	1.280	1.055	1.056
	2	5.074	5.096	3.299	3.308	1.589	1.591
	3	9.726	9.794	4.218	4.220	2.595	2.598
200.0	1	2.278	2.285	1.282	1.282	1.056	1.056
	2	5.104	5.118	3.323	3.328	1.593	1.593
	3	9.898	9.942	4.222	4.223	2.604	2.605
∞	1	2.312	2.312	1.286	1.286	1.056	1.056
	2	5.142	5.142	3.351	3.351	1.596	1.596
	3	10.01	10.01	4.225	4.225	2.613	2.613

† : $\beta(y) = \beta_0 \sin(y)$, †† : $\beta(y) = \beta_0$

the aspect ratios and rotational stiffness parameters. To compare the values of the plates having nonuniform elastic constraints with those in the case of uniform constraints, the maximum difference between them becomes 7% for the aspect ratio $\lambda' = 2.5$.

4. CONCLUSIONS

In this paper, vibration and buckling of rectangular plates with nonuniform elastic constraints in rotation have been investigated using the spline strip method.

The following conclusions have been obtained.

(1) The present approach can deal with the arbitrary distribution of elastic constraints against rotation.

(2) The values of rectangular plates calculated in the case of half sine distribution of elastic constraints in rotation are less than those of uniform constraints.

(3) The effect of nonuniform elastic constraints on the results is concerned with aspect ratios and rotational stiffness parameters of constraints.

Acknowledgement—The results presented in this paper were obtained in the course of a research programme, Grant No. 59750362, supported by the Ministry of Education, Science and Culture of the Japanese Government.

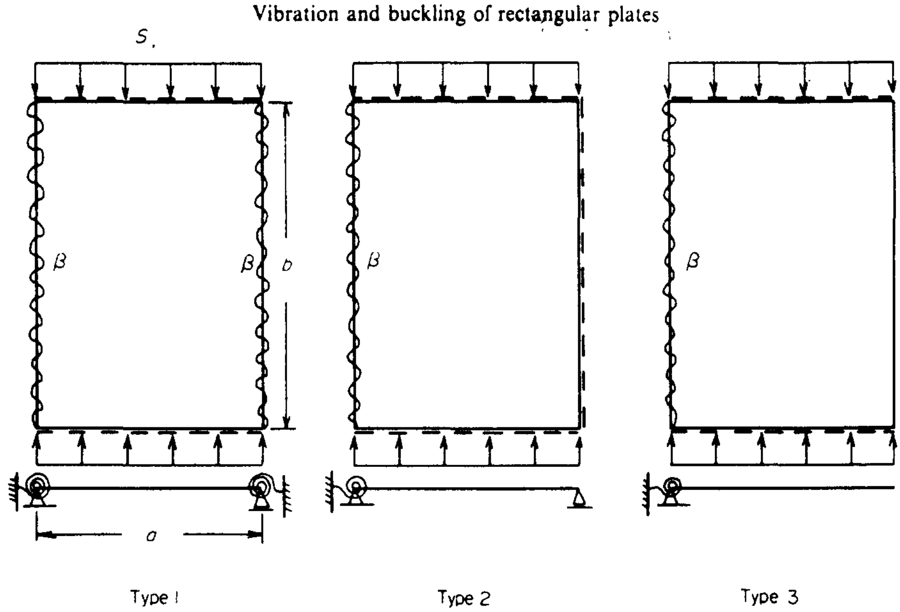

Fig. 2. Different plate boundary conditions considered.

Table 4. Buckling load parameters, k^* of rectangular plates with opposite edges elastically restrained in rotation, subjected to uniform compression S_y; $M_x = 8$, $k - 1 = 4$, $r = 15$ and $\nu = 0.3$

$\kappa=\beta_0 b/D$	Aspect ratio, $\lambda' = b/a$									
	0.5		1.0		1.5		2.0		2.5	
	◠	▭	◠	▭	◠	▭	◠	▭	◠	▭
0.0	6.250	6.250	4.000	4.000	4.340	4.340	4.000	4.000	4.134	4.134
0.1	6.267	6.270	4.034	4.040	4.351	4.355	4.014	4.020	4.142	4.146
1.0	6.404	6.428	4.320	4.373	4.440	4.485	4.133	4.194	4.206	4.243
5.0	6.788	6.847	5.250	5.414	4.772	4.939	4.592	4.833	4.454	4.617
10.0	7.032	7.096	5.963	6.168	5.084	5.327	5.040	5.342	4.708	4.979
20.0	7.262	7.317	6.746	6.953	5.506	5.799	5.429	5.705	5.093	5.485
40.0	7.438	7.476	7.183	7.317	5.972	6.255	5.792	6.096	5.594	6.059
60.0	7.511	7.541	7.304	7.418	6.225	6.476	6.013	6.303	5.910	6.287
80.0	7.551	7.575	7.378	7.476	6.386	6.607	6.163	6.431	6.108	6.415
100.0	7.577	7.597	7.428	7.491	6.497	6.694	6.271	6.518	6.216	6.504
200.0	7.631	7.643	7.544	7.597	6.763	6.888	6.550	6.720	6.507	6.718
∞	7.691	7.691	7.691	7.691	7.116	7.116	6.972	6.972	6.999	6.999

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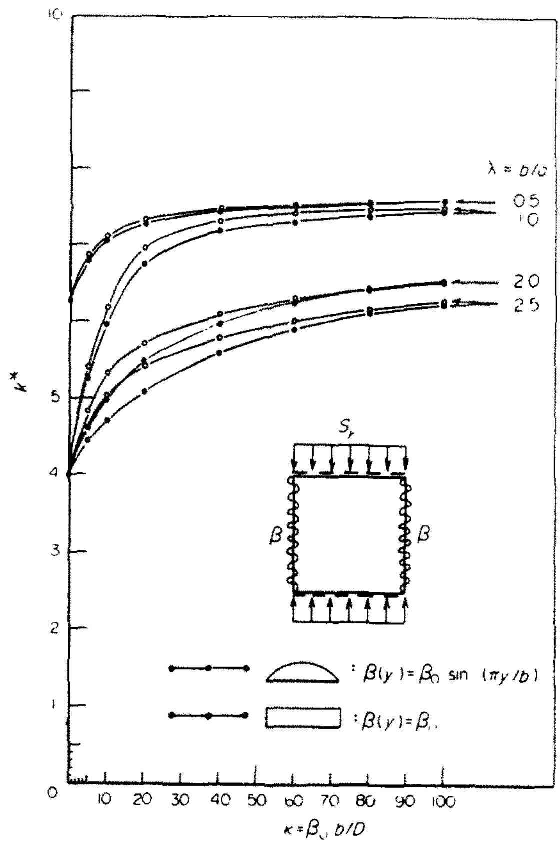
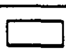
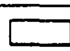
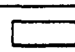
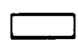


Fig. 3. The effect of nonuniformity of elastic constraints in rotation on the buckling load parameters, k^*.

Table 5. Buckling load parameters, k^* of rectangular plates with opposite edges simply supported and elastically restrained in rotation, subjected to uniform compression S_y; $M_x = 8$, $k - 1 = 4$, $r = 15$ and $\nu = 0.3$

κ=β₀b/D	Aspect ratio, λ' = b/a									
	0.5		1.0		1.5		2.0		2.5	
	⌓	▭	⌓	▭	⌓	▭	⌓	▭	⌓	▭
0.0	6.250	6.250	4.000	4.000	4.340	4.340	4.000	4.000	4.134	4.134
0.1	6.258	6.260	4.017	4.020	4.345	4.348	4.007	4.010	4.138	4.140
1.0	6.325	6.337	4.157	4.182	4.390	4.411	4.066	4.096	4.170	4.188
5.0	6.501	6.527	4.577	4.646	4.548	4.624	4.286	4.395	4.291	4.367
10.0	6.604	6.630	4.866	4.944	4.688	4.791	4.488	4.646	4.410	4.529
20.0	6.696	6.717	5.155	5.227	4.865	4.980	4.758	4.944	4.583	4.740
40.0	6.762	6.776	5.387	5.440	5.046	5.148	5.050	5.227	4.791	4.960
60.0	6.789	6.800	5.487	5.528	5.138	5.225	5.206	5.361	4.915	5.073
80.0	6.803	6.812	5.542	5.576	5.194	5.269	5.305	5.425	4.997	5.142
100.0	6.813	6.820	5.578	5.607	5.232	5.297	5.369	5.455	5.056	5.188
200.0	6.832	6.836	5.654	5.671	5.320	5.360	5.467	5.524	5.205	5.294
∞	6.853	6.853	5.740	5.740	5.431	5.431	5.606	5.606	5.423	5.423

Table 6. Buckling load parameters, k^* of rectangular plates with opposite edges elastically restrained in rotation and free (Type 3), subjected to uniform compression S_y; $M_x = 8$, $k - 1 = 4$, $r = 15$ and $\nu = 0.3$

$\kappa=\beta_0 b/D$	Aspect ratio, $\lambda'=b/a$									
	0.5		1.0		1.5		2.0		2.5	
										
0.0	4.356	4.356	1.402	1.402	0.8578	0.8578	0.6681	0.6681	0.5806	0.5806
0.1	4.357	3.358	1.405	1.405	0.8622	0.8629	0.6738	0.6748	0.5875	0.5887
1.0	4.370	4.373	1.430	1.434	0.8982	0.9047	0.7207	0.7294	0.6455	0.6563
5.0	4.403	4.407	1.498	1.509	1.005	1.022	0.8679	0.8930	0.8356	0.8694
10.0	4.421	4.426	1.541	1.552	1.077	1.096	0.9753	1.005	0.9829	1.026
20.0	4.437	4.440	1.581	1.590	1.149	1.166	1.088	1.117	1.146	1.190
40.0	4.448	4.450	1.610	1.617	1.205	1.218	1.183	1.205	1.205	1.295
60.0	4.453	4.454	1.622	1.628	1.230	1.240	1.225	1.243	1.292	1.319
80.0	4.455	4.456	1.629	1.633	1.243	1.251	1.249	1.264	1.309	1.333
100.0	4.457	4.458	1.633	1.637	1.252	1.259	1.264	1.277	1.321	1.342
200.0	4.460	4.460	1.643	1.644	1.270	1.274	1.298	1.305	1.349	1.362
∞	4.463	4.463	1.653	1.653	1.291	1.291	1.338	1.388	1.385	1.385

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